

On Dynamic-order Fractional Dynamic System

HongGuang Sun^{a,b}, Hu Sheng^{b,c}, YangQuan Chen^b, Wen Chen^{a,*}, and ZhongBo Yu^d

a. Institute of Soft Matter Mechanics,

Department of Engineering Mechanics, Hohai University,

No.1 XiKang Road, Nanjing, Jiangsu 210098, China

b. Center for Self-Organizing and Intelligent Systems (CSOIS),

Department of Electrical and Computer Engineering,

Utah State University, 4160 Old Main Hill, Logan, UT 84322-4160, USA

c. Department of Electronic Engineering,

Dalian University of Technology, Dalian 116024, China

d. College of Hydrology and Water Resources, Hohai University,

No.1 XiKang Road, Nanjing, Jiangsu 210098, China

** Corresponding author: chenwen@hhu.edu.cn*

(Dated: January 13, 2013)

Abstract

Multi-system interaction is an important and difficult problem in physics. Motivated by the experimental result of an electronic circuit element “Fractor”, we introduce the concept of dynamic-order fractional dynamic system, in which the differential-order of a fractional dynamic system is determined by the output signal of another dynamic system. The new concept offers a comprehensive explanation of physical mechanism of multi-system interaction. The properties and potential applications of dynamic-order fractional dynamic systems are further explored with the analysis of anomalous relaxation and diffusion processes.

PACS numbers: 66.10.C-, 05.10.Gg, 02.60.Cb

Fractional dynamic system has been focused by physicists and mathematicians over the last decades, and has received great success in the analysis of anomalous diffusion [1–4], viscoelastic rheology [5, 6], control systems [7, 8], complex networks [9], wave dissipation in human tissue and electrochemical corrosion process [10], etc. [11–13].

In the past several years, various physical applications have given birth to the variable-order fractional dynamic system [14–16]. However, the multi-system interaction must be considered in the physical mechanism analysis of the variable-order fractional dynamic system and its applications. Especially, the behavior of a dynamic system may change with the evolution of other dynamic systems in multi-system physical processes. How to characterize the interaction effect between these dynamic systems? Establishing a system of equations which includes intricate interaction terms, will cause great difficulties in modeling and computation. Even worse, since they may miss capturing the critical physical mechanism of the considered problems, the established model will produce incorrect results which greatly deviate from experimental results or field measurement data. Meanwhile, researchers have confirmed that the differential orders in some fractional dynamic systems are non-constant and are often functions of other variables or system outputs [16, 17]. For instance, Glöckle and Nonnenmacher have found that the differential order of proteins relaxation is a function of temperature [18]. Therefore, in order to exploit the physical mechanism of a variable-order fractional dynamic system, another dynamic system usually should be considered. For simplicity, we name this type of variable-order fractional dynamic system as dynamic-order fractional dynamic system.

The purpose of this letter is to make an innovative study of dynamic-order fractional dynamic systems. Since previous studies have indicated that the differential-order is a critical factor in the variable-order fractional dynamic system, it is necessary to analyze the physical mechanism about how environmental factors or system variables influence the system's differential order. From the analysis of fractional dynamic system, especially the variable-order dynamic system, it has been found the fractional differential order of one dynamic system usually stems from another dynamic system [16–18]. Hence, in many fractional systems, the differential order should be called dynamic-order. The comprehensive study of dynamic-order system will give us new understanding about multi-system interaction.

Firstly, we introduce the motivation of dynamic-order fractional system via an experiment of Fractor. This experiment will give us a basic understanding of dynamic-order fractional

dynamic systems.

A Fractor is a two lead passive electronic circuit element similar to a resistor or capacitor, exhibiting a non-integer order power-law impedance versus frequency [19]. The Fractor is proven effective as a feedback element in control systems for real-world applications such as temperature control, robotics, etc. [20]. The prototype Fractor is made by hand and not much larger than typical through-hole capacitors and the typical unit is 3.5 cm on a side and about 1.0 cm thick, as shown in Fig. 1 (Left). The impedance behavior of the Fractor can be accurately modeled by the following formula which is achieved by Laplace transform of the fractional order operator $Z_{Fractor}(\omega) = K/(j\omega\tau)^\lambda$, K is the impedance magnitude at calibration frequency ($\omega_c = 1/\tau$); $\lambda \in (0, 1)$ is the fractional exponent and ω is the frequency [19, 21]. From the previous study of the Fractor, it has been confirmed that the order of the Fractor may change with the temperature and its internal material structure. Several experimental results have implied that, temperature can influence the derivative order or integral order which will determine the capacity of the equipment [18, 19]. In our experiment, we intend to give a clear relationship between the temperature and the fractional order of the Fractor. In this experiment, temperature is controlled by the equipment of Quanser Heatflow Experiment (HFE) [19]. The evolution process of temperature is a typical heat transfer process. In this heat transfer system, the instantaneous temperature value of the Fractor is the output signal we intend to obtain. Based on the previous experimental observation, we rewrite the above formula as

$$Z_{Fractor}(\omega) = \frac{K}{(j\omega\tau)^{\lambda(T)}}, \quad (1)$$

where $\lambda(T) \in (0, 1)$ is the fractional exponent, T is the temperature.

In the experiment, we firstly placed the Fractor in the HFE unit, then the temperature of the Fractor was measured by three temperature sensors. Next, the order of the Fractor was calculated by an HP DSA and fitted via the expression (1). Finally, The relationship between the temperature and the order of the Fractor was obtained as shown in Fig. 1 (Right). The experimental details were stated in [20]. It is observed that the order of the Fractor is an approximately linear function of the temperature. Since the order of the Fractor changes with the temperature which is the output signal of the heat transfer system in the Fractor, this system is a typical dynamic-order system.

Next, we discuss the dynamic-order fractional dynamic system from the viewpoint of

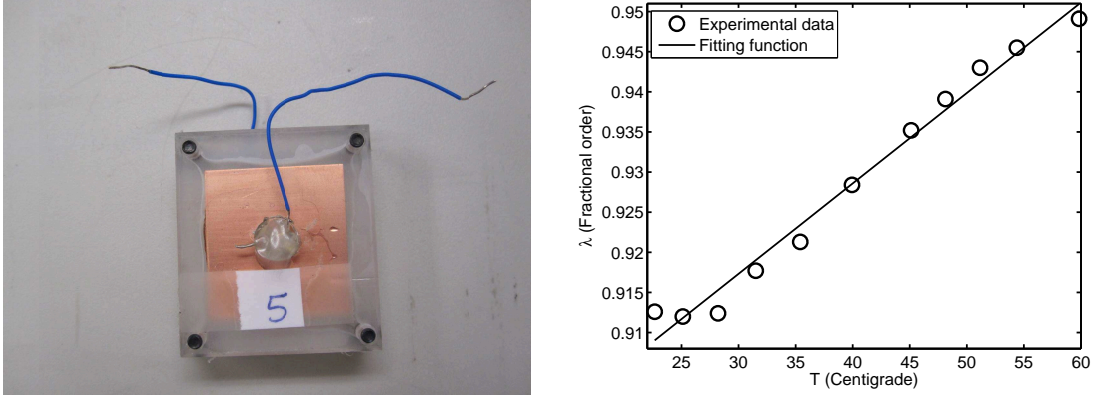


FIG. 1: Left: Photo of a sample hand-made prototype Fractor. The detailed preparation and description of the Fractor can be found in [19]. Right: The fractional order evolution data of the Fractor. In this temperature range, the linear fitting function is $\lambda(T) = p_1T + p_2$, where $p_1 = 0.001127 \pm 0.000116$, $p_2 = 0.8835 \pm 0.0048$.

variable-order fractional dynamic system. A representative definition of variable-order fractional derivative in Caputo sense can be stated as follows [22, 23]

$${}^C D_{0+}^{\alpha(Z)} f(t) = \frac{1}{\Gamma(1 - \alpha(Z))} \int_0^t \frac{f'(\tau) d\tau}{(t - \tau)^{\alpha(Z)}}, \quad 0 < \alpha(Z) < 1, \quad (2)$$

where Z denotes certain system variable or other independent variable, which govern the differential-order of interested dynamic-order system. $\alpha(Z)$ is a function of the independent variable Z and $\Gamma()$ is the Gamma function. In some variable-order fractional dynamic systems, the variable-order is a function of certain variable, such as temperature or concentration, while the order-influencing variable usually can be regarded as output signal of another dynamic system.

In engineering practice, we usually encounter the phenomena of the multi-system interaction, such as temperature field, stress field and electromagnetic field coupled together with the concerned dynamic system under investigation. From the traditional approach, when considering the coupling effect of different systems or some multi-scale physical processes, we are accustomed to employing a system of differential equations and they have received great success in the past decades [24]. For example, when modeling the tracer transport in a fractured granite, Reimus et al. [25] established a convection-diffusion equation in fracture and a diffusion equation in matrix, in which the intricate interaction is reflected by an additional term in the convection-diffusion equation in fracture scale. However, in the recent

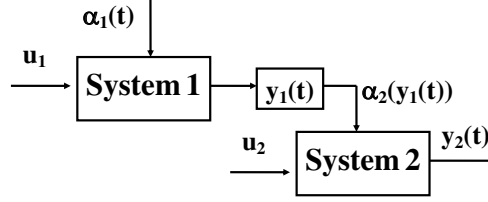


FIG. 2: An illustration of a dynamic-order fractional dynamic system. In this schematic, u_1 is the input of system 1 and $\alpha_1(t)$ is the differential order, $y_1(t)$ is the output signal. $\alpha_2(y_1(t))$ denotes the differential-order of system 2, u_2 and $y_2(t)$ represent the input and output signals of system 2.

decades, experimental and theoretical studies have implied that the dynamic-order system approach can be employed to reflect the interaction effect in some real-world multi-system problems [16, 19]. The link between these systems may be established via differential order of each dynamic system, which can be visually illustrated by Fig. 2.

From the viewpoint of dynamic-order system, when investigating some multi-system problems, we can establish the following generalized form of dynamic-order fractional dynamic systems

$$\left\{ \begin{array}{l} \frac{d^{\alpha_1(X_n, X_{n-1}, \dots, X_1, t)} X_1}{dt^{\alpha_1(X_n, X_{n-1}, \dots, X_1, t)}} = g_1(X_1, t), \\ \frac{d^{\alpha_2(X_n, X_{n-1}, \dots, X_1, t)} X_2}{dt^{\alpha_2(X_n, X_{n-1}, \dots, X_1, t)}} = g_2(X_2, t), \\ \vdots \\ \frac{d^{\alpha_n(X_n, X_{n-1}, \dots, X_1, t)} X_n}{dt^{\alpha_n(X_n, X_{n-1}, \dots, X_1, t)}} = g_n(X_n, t). \end{array} \right. \quad (3)$$

The most important feature of the above generalized form is that the intricate interaction between dynamic systems has been represented by the differential order of each sub-system.

In the following, we further illustrate the dynamic-order fractional dynamic system through two cases.

Case 1. We firstly consider the relaxation system, which has been widely applied in energy dissipation, viscoelasticity and rheology, etc. [22, 26, 27]. To characterize the relaxation process, in which the relaxation pattern changes with control parameters or other variables,

the following variable-order fractional differential equation may be employed

$$\begin{cases} {}^C D_t^{\alpha(Z)} x(t) = -Bx(t) + f(t), 0 < \alpha(Z) < 1, \\ x(0) = 1, \end{cases} \quad (4)$$

where B is the relaxation coefficient and Z is an independent variable.

In engineering fields, fuzzy systems have extensive applications in control and system parameter calibration, and have become an efficient tool to exploit clear conclusion from uncertain information and incomplete data [28–30]. Thereby, we consider the dynamic system (4) in which the variable-order is governed by the following fuzzy dynamic system. Here, we select the Takagi-Sugeno (T-S) fuzzy system as an example for illustration purpose. The T-S fuzzy system obeys the following rule [31]:

Plant Rule i : If $s_1(t)$ is μ_{i1} and \dots and $s_p(t)$ is μ_{ip} , then

$$y(t) = A_i y_0(i), i = 1, 2, \dots, N, \quad (5)$$

where μ_{ij} is the fuzzy set and N is the number of IF-THEN rules; $s_1(t), \dots, s_p(t)$ are the premise variables. Then the final output of T-S fuzzy system is inferred as follows:

$$\begin{cases} \dot{y}(t) = \sum_{i=1}^N h_i(y(t)) A_i y(t), \\ y(0) = 1.0. \end{cases} \quad (6)$$

In this illustrative case, we assume $N = 2$ and the fuzzy membership $h_1(y(t)) = \frac{1}{2} - \frac{1}{2}y(t)$, $h_2(y(t)) = \frac{1}{2} + \frac{1}{2}y(t)$.

If $A_1 = 0, A_2 = -1$, then the exact solution of (6) is

$$y(t) = \frac{1}{2e^{t/2} - 1}. \quad (7)$$

For numerical simulation, we assume that the relationship between the output of fuzzy system (6) and variable-order of fractional dynamic system (4) can be stated as follows

$$\alpha(Z) = \alpha(t) = 1.0 - 0.2y(t). \quad (8)$$

The evolution curve of variable-order $\alpha(t)$ and the corresponding numerical result of the relaxation system (4) are shown in Fig. 3.

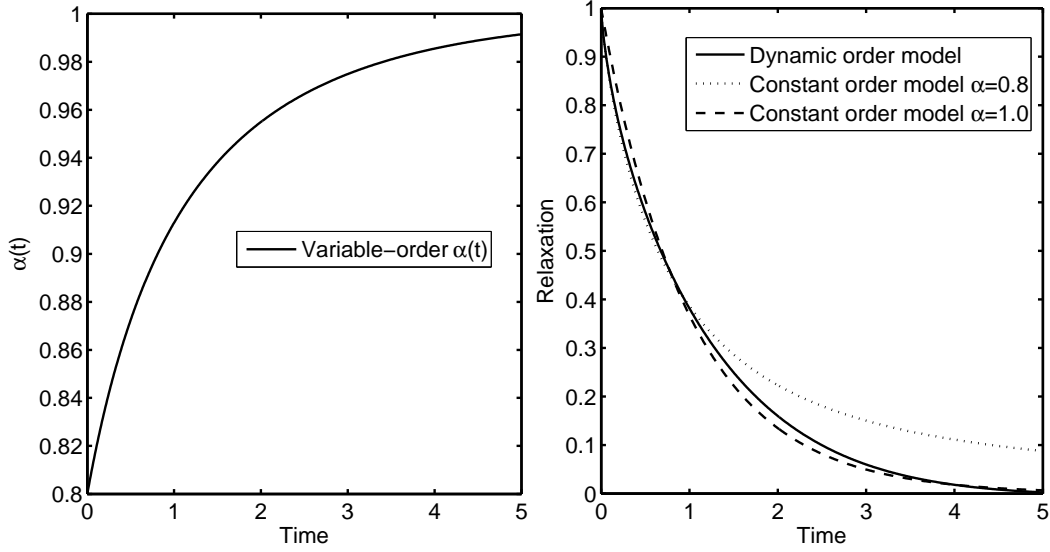


FIG. 3: Left: The curve of the variable-order $\alpha(t)$ originated from the fuzzy system (6). Right: The relaxation curve of the system (4) with the dynamic-order $\alpha(t)$ given in (8) and $B = 1.0$.

From the observation of Fig. 3, the system (4) exhibits accelerating relaxation behavior with the variable-order (8). Since fuzzy systems have been regarded as efficient tools to tackle real-world engineering problems, this case implies that, fuzzy systems may be an important method to characterize the evolution behavior of variable-order in fractional dynamic system.

Case 2. We further consider the variable-order fractional diffusion system, in which the differential order is determined by a fractional order dynamic system. It means that the differential order in the considered fractional diffusion system relates to the output of another fractional dynamic system. It is a more general form to investigate the dynamic-order system. The considered variable-order fractional diffusion system is stated as

$$\begin{cases} {}^C D_t^{\alpha(T)} u(x, t) = K \frac{\partial^2 u(x, t)}{\partial x^2} + q(x, t), \\ u(x, 0) = \sin(x), x \in [0, L], \\ u(0, t) = u(L, t) = 0, t \in [0, M], \end{cases} \quad (9)$$

where $K > 0$ is the generalized diffusion coefficient, $u(x, t)$ is concentration, mass or other quantities of interest, $q(x, t)$ is a source term and $\alpha(T)$ is a function of temperature.

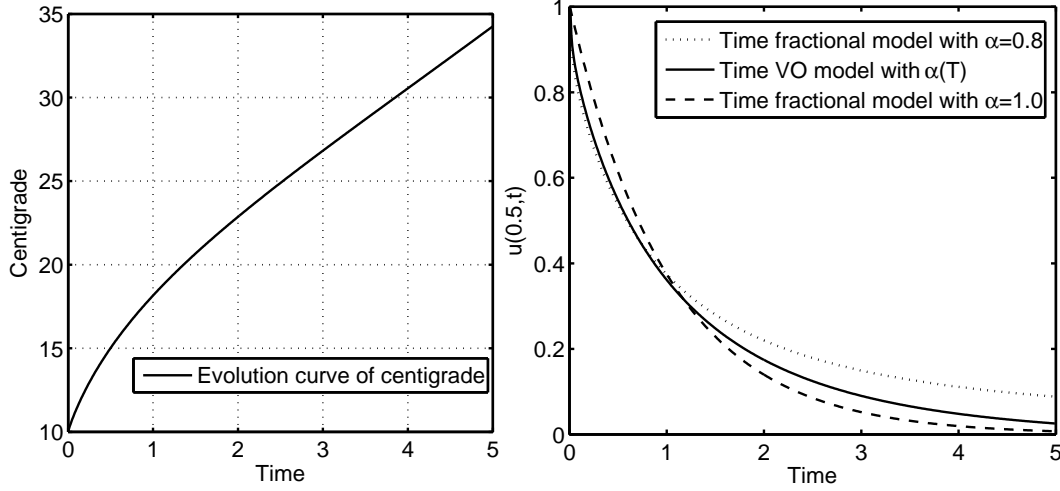


FIG. 4: Left: The evolution curve of the temperature (10) with $\beta = 0.9$. In numerical simulation, the time step is 0.01. Right: The diffusion curve of the system (9) with $L = 1.0$, $M = 5.0$, $x = 0.5$, $q(x, t) = 0$ and $K = 0.1$, the differential order is (11).

We suppose the evolution process of temperature can be characterized by

$$\begin{cases} {}^C D_t^\beta T(t) = 0.1T(t) + 10/(1.3t + 1), \\ T(0) = 10, \end{cases} \quad (10)$$

where $\beta \in (0, 1]$ is the fractional order derivative. Then, we assume the relationship between the differential order (9) and the temperature is

$$\alpha(T) = 0.8 + 0.005T. \quad (11)$$

It indicates that the diffusion process characterized by (9) is an accelerating subdiffusion process. The trajectory of $\alpha(t)$ and the diffusion curve of system (9) at $x = 0.5$ are drawn in Fig. 4.

This example shows that the heat transfer process with result of temperature increasing, causes the accelerating behavior of considered diffusion process. In engineering situations, this model can offer an effective tool to explore physical mechanism of real-world diffusion related dynamic processes.

Finally, we should note that, though we have made a first attempt to introduce the concept of dynamic-order fractional dynamic system, in which the multi-system interaction

is delivered by the differential order of each system. The further investigations on physical mechanism and application potentials are deserved.

H. G. Sun thank G. W. Bohannan, X. N. Song and the AFC reading group meeting in CSOIS, Utah State University for discussions. The work is supported by 2010CB832702, 201101014 and 2010B19114.

-
- [1] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
 - [2] G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002).
 - [3] R. L. Magin, O. Abdullah, D. Baleanu, and X. J. Zhou, J. Magn. Reson. **190(2)**, 255 (2008).
 - [4] I. M. Sokolov and J. Klafter, Phys. Rev. Lett. **97**, 140602 (2006).
 - [5] A. Hernandez-Jimenez, J. Hernandez-Santiago, A. Macias-Garcia, and J. Sanchez-Gonzalez, Polym. Test. **21(3)**, 325 (2002).
 - [6] P. Yang, Y. Lam, and K. Q. Zhu, J. Non-Newtonian Fluid Mech. **165**, 88 (2010).
 - [7] Y. Q. Chen and K. L. Moore, IEEE Trans. Circuits Syst. I **49(3)**, 363 (2002).
 - [8] I. Podlubny, IEEE Trans. Automat. Contr. **44(1)**, 208 (1999).
 - [9] B. J. West, E. L. Geneston, and P. Grigolini, Phys. Rep. **468**, 1 (2008).
 - [10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (North-Holland, Boston, 2006).
 - [11] D. Baleanu and S. I. Muslih, Phys. Scripta **72**, 119 (2005).
 - [12] M. Magdziarz, A. Weron, and J. Klafter, Phys. Rev. Lett. **101**, 210601 (2008).
 - [13] A. I. Saichev and G. M. Zaslavsky, Chaos **7(4)**, 753 (1997).
 - [14] S. G. Samko, J. Anal. Math. **21**, 213 (1995).
 - [15] C. F. Lorenzo and T. T. Hartley, Nonlinear Dyn. **29**, 57 (2002).
 - [16] H. G. Sun, W. Chen, and Y. Q. Chen, Phys. A **388**, 4586 (2009).
 - [17] D. Ingman, J. Suzdalnitsky, and M. Zeifman, J. Appl. Mech. **67**, 383 (2000).
 - [18] W. G. Glöckle and T. F. Nonnenmacher, Biophys. J. **46**, 787 (1995).
 - [19] G. W. Bohannan, J. Vib. Control **14**, 1487 (2008).
 - [20] H. Sheng, H. G. Sun, C. Coopmans, Y. Q. Chen, and G. W. Bohannan, Eur. Phys. J. Special Topics **193**, 93 (2011).
 - [21] I. S. Jesus, J. A. T. Machado, and J. B. Cunha, J. Vib. Control **14**, 1389 (2008).

- [22] C. F. M. Coimbra, Ann. Phys. (Leipzig) **12(11-12)**, 692 (2003).
- [23] H. G. Sun, W. Chen, H. Sheng, and Y. Q. Chen, Phys. Lett. A **374**, 906 (2010).
- [24] S. A. Meier, M. A. Peter, and M. Böhm, Comput. Mater. Sci. **39**, 29 (2007).
- [25] P. Reimus, G. Pohll, T. Mihevc, J. Chapman, M. Haga, S. K. B.Lyles, R. Niswonger, and P. Sanders, Water Resour. Res. **39(12)**, 1356 (2003).
- [26] R. L. Bagley and P. J. Torvik, J. Rheol. **27(3)**, 201 (1983).
- [27] R. Metzler and J. Klafter, J. of Non-Crystalline Solids **305**, 81 (2002).
- [28] H. O. Wang, K. Tanaka, and M. F. Griffin, IEEE Trans. on Fuzzy Syst. **4(1)**, 14 (1996).
- [29] B. Kosko and J. C. Burgess, J. Acoust. Soc. Am. **103(6)**, 3131 (1998).
- [30] H. L. Wang, S. Kwong, Y. C. Jin, W. Wei, and K. F. Man, Fuzzy Set Syst. **149(1)**, 149 (2005).
- [31] X. N. Song, S. Y. Xu, and H. Shen, Informat. Sci. **178**, 4341 (2008).